Homework 1

This assignment is due by **11:59pm on Friday 9/6** on gradescope. You should start a new page for each problem (if a problem has multiple parts, the parts can be completed on the same page). After completing the assignment, take (nonblurry) photos/scans of your solutions, save them as a single pdf, and upload your file to gradescope (there are many free apps that allow you to scan and save documents as a pdf; e.g. camscanner). If you have not used gradescope before, see `gradescope.com/help`.

You may collaborate with your peers on homework assignments and to discuss points of confusion with the instructor and/or TAs. However, you must write your own solutions independently and you may not look at or copy any one else's solutions.

Note: This assignment covers material from lectures 8/26-9/4.

1. Let m, n, p be distinct positive integers and $m > 1$.

   - Let A be an $m \times n$ matrix
   - Let B and C be $n \times p$ matrices
   - Let D be a $p \times m$ matrix
   - Let $\mathbf{v} \in \mathbb{R}^n$

   For each of the following expressions, describe what kind of object is produced (including the dimension if the object is a vector or the number of rows and columns if the object is a matrix), or if the expression is not well- defined, explain why.

   (a) $A\mathbf{v}$

   (b) $B\mathbf{v}$

   (c) $\mathbf{v}A$

   (d) AB

   (e) ABC

   (f) ABD

   (g) $A + B$

   (h) $A(B + C)$

2. Consider the vectors $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

   (a) Compute the linear combination $2\mathbf{v} + 3\mathbf{w}$.

   (b) Sketch $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^2$. Then sketch the linear combination vector $2\mathbf{v} + 3\mathbf{w}$ by first scaling $\mathbf{v}$ and $\mathbf{w}$, and then using the parallelogram rule.

(c) Set up and solve a system of equations to answer the following question: Is $\begin{bmatrix}3\\4\end{bmatrix}$ a linear combination of $\mathbf{v}$ and $\mathbf{w}$? If so, what are the weights?

3. Consider the linear transformation T given by the 2×2 matrix $A = \begin{bmatrix}2 & 0\\1 & 1\end{bmatrix}$.

   (a) Plot the vectors $\begin{bmatrix}a\\b\end{bmatrix}$ where $0 \leq a \leq 3$ and $0 \leq b \leq 3$ in $\mathbb{R}^2$. Here, instead of drawing the arrows, shade in the region populated by the vectors.

   (b) Plot the vectors $T\Big(\begin{bmatrix}a\\b\end{bmatrix}\Big)$ where $0 \leq a \leq 3$ and $0 \leq b \leq 3$ in $\mathbb{R}^2$. Here, instead of drawing the arrows, shade in the region populated by the vectors.

   (c) Let S be the linear transformation given by $B = \begin{bmatrix}-1 & 1 & 2\\2 & 1 & -1\end{bmatrix}$. Compute the matrices associated to the linear transformations $S \circ T$ and $T \circ S$ (if they exist).

4. Consider the map $T : \mathbb{R} \to \mathbb{R}$ given by $T(x) = mx + b$, where $m, b \in \mathbb{R}$ and $m \neq 0$.

   (a) Show that if $b = 0$, then T is a linear transformation.

   (b) Show that if $b \neq 0$, then T is not a linear transformation.[1]

5. Solve the following system by using Gaussian-Jordan elimination to put the associated augmented matrix in RREF. Be sure to explain your steps.

$$\begin{aligned} 2x + 3y + z &= 8\\ 4x + 7y + 5z &= 20\\ -2y + 2z &= 0 \end{aligned}$$

6. Recall the associative property for vector addition: $\mathbf{v} + (\mathbf{w} + \mathbf{u}) = (\mathbf{v} + \mathbf{w}) + \mathbf{u}$ for all $\mathbf{v}, \mathbf{w}, \mathbf{u} \in \mathbb{R}^n$.

   (a) Prove this property.

   (b) Symbolically prove this property using Julia for three symbolic vectors in $\mathbb{R}^5$. Take a picture of screenshot of your code and output and add it as the last page of your solutions.

[1] If $b \neq 0$, then f is called an *affine transformation*